AP® Calculus AB
Free-Response Scoring Guidelines

Question 1

The rate at which raw sewage enters a treatment tank is given by $E(t) = 850 + 715\cos\left(\frac{\pi t^2}{9}\right)$ gallons per hour for $0 \le t \le 4$ hours. Treated sewage is removed from the tank at the constant rate of 645 gallons per hour. The treatment tank is empty at time $t = 0$.

(a) How many gallons of sewage enter the treatment tank during the time interval $0 \le t \le 4$? Round your answer to the nearest gallon.

(b) For $0 \le t \le 4$, at what time t is the amount of sewage in the treatment tank greatest? To the nearest gallon, what is the maximum amount of sewage in the tank? Justify your answers.

(c) For $0 \le t \le 4$, the cost of treating the raw sewage that enters the tank at time t is $(0.15 - 0.02t)$ dollars per gallon. To the nearest dollar, what is the total cost of treating all the sewage that enters the tank during the time interval $0 \le t \le 4$?

(a) $\int_0^4 E(t)\,dt \approx 3981$ gallons

$$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$$

(b) Let $S(t)$ be the amount of sewage in the treatment tank at time t. Then $S'(t) = E(t) - 645$ and $S'(t) = 0$ when $E(t) = 645$. On the interval $0 \le t \le 4$, $E(t) = 645$ when $t = 2.309$ and $t = 3.559$.

t (hours)	amount of sewage in treatment tank
0	0
2.309	$\int_0^{2.309} E(t)\,dt - 645(2.309) = 1637.178$
3.559	$\int_0^{3.559} E(t)\,dt - 645(3.559) = 1228.520$
4	$3981.022 - 645(4) = 1401.022$

The amount of sewage in the treatment tank is greatest at $t = 2.309$ hours. At that time, the amount of sewage in the tank, rounded to the nearest gallon, is 1637 gallons.

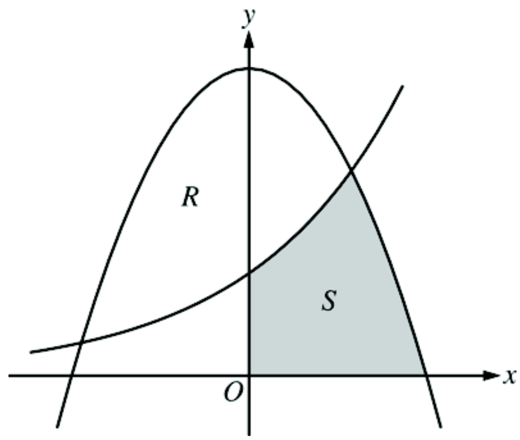
$$4: \begin{cases} 1: \text{sets } E(t) = 645 \\ 1: \text{identifies } t = 2.309 \text{ as} \\ \quad \text{a candidate} \\ 1: \text{amount of sewage at } t = 2.309 \\ 1: \text{conclusion} \end{cases}$$

(c) Total cost $= \int_0^4 (0.15 - 0.02t)E(t)\,dt = 474.320$

The total cost of treating the sewage that enters the tank during the time interval $0 \le t \le 4$, to the nearest dollar, is \$474.

$$3: \begin{cases} 1: \text{integrand} \\ 1: \text{limits} \\ 1: \text{answer} \end{cases}$$

Question 2



Let R and S in the figure above be defined as follows: R is the region in the first and second quadrants bounded by the graphs of $y = 3 - x^2$ and $y = 2^x$. S is the shaded region in the first quadrant bounded by the two graphs, the x-axis, and the y-axis.

(a) Find the area of S.

(b) Find the volume of the solid generated when R is rotated about the horizontal line $y = -1$.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is an isosceles right triangle with one leg across the base of the solid. Write, but do not evaluate, an integral expression that gives the volume of the solid.

$3 - x^2 = 2^x$ when $x = -1.63658$ and $x = 1$ Let $a = -1.63658$	
(a) Area of $S = \int_0^1 2^x \, dx + \int_1^{\sqrt{3}} \left(3 - x^2\right) dx$ $= 2.240$	$3: \begin{cases} 1: \text{integrands} \\ 1: \text{limits} \\ 1: \text{answer} \end{cases}$
(b) Volume $= \pi \int_a^1 \left(\left(3 - x^2 + 1\right)^2 - \left(2^x + 1\right)^2\right) dx$ $= 63.106$ or 63.107	$4: \begin{cases} 2: \text{integrand} \\ 1: \text{limits and constant} \\ 1: \text{answer} \end{cases}$
(c) Volume $= \dfrac{1}{2} \int_a^1 \left(3 - x^2 - 2^x\right)^2 dx$	$2: \begin{cases} 1: \text{integrand} \\ 1: \text{limits and constant} \end{cases}$

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Question 3

t (minutes)	0	4	8	12	16
$H(t)$ (°C)	65	68	73	80	90

The temperature, in degrees Celsius $(°\text{C})$, of an oven being heated is modeled by an increasing differentiable function H of time t, where t is measured in minutes. The table above gives the temperature as recorded every 4 minutes over a 16-minute period.

(a) Use the data in the table to estimate the instantaneous rate at which the temperature of the oven is changing at time $t = 10$. Show the computations that lead to your answer. Indicate units of measure.

(b) Write an integral expression in terms of H for the average temperature of the oven between time $t = 0$ and time $t = 16$. Estimate the average temperature of the oven using a left Riemann sum with four subintervals of equal length. Show the computations that lead to your answer.

(c) Is your approximation in part (b) an underestimate or an overestimate of the average temperature? Give a reason for your answer.

(d) Are the data in the table consistent with or do they contradict the claim that the temperature of the oven is increasing at an increasing rate? Give a reason for your answer.

(a) $H'(10) \approx \dfrac{H(12) - H(8)}{12 - 8} = \dfrac{80 - 73}{4} = \dfrac{7}{4}\,°\text{C}/\text{min}$

$2 : \begin{cases} 1 : \text{difference quotient} \\ 1 : \text{answer with units} \end{cases}$

(b) Average temperature is $\dfrac{1}{16}\displaystyle\int_0^{16} H(t)\,dt$

$\displaystyle\int_0^{16} H(t)\,dt \approx 4 \cdot (65 + 68 + 73 + 80)$

Average temperature $\approx \dfrac{4 \cdot 286}{16} = 71.5°\text{C}$

$3 : \begin{cases} 1 : \dfrac{1}{16}\displaystyle\int_0^{16} H(t)\,dt \\ 1 : \text{left Riemann sum} \\ 1 : \text{answer} \end{cases}$

(c) The left Riemann sum approximation is an underestimate of the integral because the graph of H is increasing. Dividing by 16 will not change the inequality, so $71.5°\text{C}$ is an underestimate of the average temperature.

1 : answer with reason

(d) If a continuous function is increasing at an increasing rate, then the slopes of the secant lines of the graph of the function are increasing. The slopes of the secant lines for the four intervals in the table are $\frac{3}{4}$, $\frac{5}{4}$, $\frac{7}{4}$, and $\frac{10}{4}$, respectively.

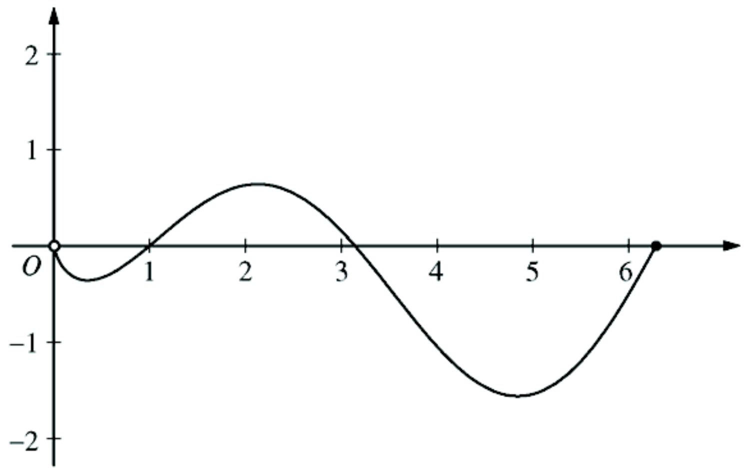
Since the slopes are increasing, the data are consistent with the claim.

OR

By the Mean Value Theorem, the slopes are also the values of $H'(c_k)$ for some times $c_1 < c_2 < c_3 < c_4$, respectively. Since these derivative values are positive and increasing, the data are consistent with the claim.

$3 : \begin{cases} 1 : \text{considers slopes of four secant lines} \\ 1 : \text{explanation} \\ 1 : \text{conclusion consistent with explanation} \end{cases}$

Question 4



Graph of f

Let f be the function given by $f(x) = (\ln x)(\sin x)$. The figure above shows the graph of f for $0 < x \le 2\pi$. The function g is defined by $g(x) = \int_1^x f(t)\,dt$ for $0 < x \le 2\pi$.

(a) Find $g(1)$ and $g'(1)$.

(b) On what intervals, if any, is g increasing? Justify your answer.

(c) For $0 < x \le 2\pi$, find the value of x at which g has an absolute minimum. Justify your answer.

(d) For $0 < x < 2\pi$, is there a value of x at which the graph of g is tangent to the x-axis? Explain why or why not.

(a) $g(1) = \int_1^1 f(t)\,dt = 0$ and $g'(1) = f(1) = 0$	$2 : \begin{cases} 1 : g(1) \\ 1 : g'(1) \end{cases}$
(b) Since $g'(x) = f(x)$, g is increasing on the interval $1 \le x \le \pi$ because $f(x) > 0$ for $1 < x < \pi$.	$2 : \begin{cases} 1 : \text{interval} \\ 1 : \text{reason} \end{cases}$
(c) For $0 < x < 2\pi$, $g'(x) = f(x) = 0$ when $x = 1,\ \pi$. $g' = f$ changes from negative to positive only at $x = 1$. The absolute minimum must occur at $x = 1$ or at the right endpoint. Since $g(1) = 0$ and $g(2\pi) = \int_1^{2\pi} f(t)\,dt = \int_1^{\pi} f(t)\,dt + \int_{\pi}^{2\pi} f(t)\,dt < 0$ by comparison of the two areas, the absolute minimum occurs at $x = 2\pi$.	$3 : \begin{cases} 1 : \text{identifies 1 and } 2\pi \text{ as candidates} \\ \qquad \text{- or -} \\ \quad \text{indicates that the graph of } g \\ \quad \text{decreases, increases, then decreases} \\ 1 : \text{justifies } g(2\pi) < g(1) \\ 1 : \text{answer} \end{cases}$
(d) Yes, the graph of g is tangent to the x-axis at $x = 1$ since $g(1) = 0$ and $g'(1) = 0$.	$2 : \begin{cases} 1 : \text{answer of “yes” with } x = 1 \\ 1 : \text{explanation} \end{cases}$

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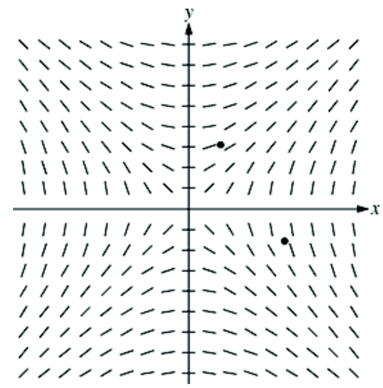
Question 5

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{x}{y}$, where $y \neq 0$.

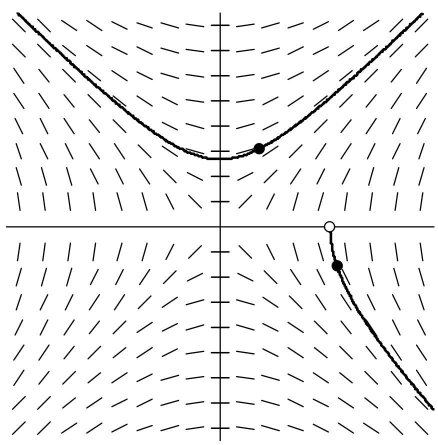
(a) The slope field for the given differential equation is shown below. Sketch the solution curve that passes through the point $(3, -1)$, and sketch the solution curve that passes through the point $(1, 2)$. (Note: The points $(3, -1)$ and $(1, 2)$ are indicated in the figure.)



(b) Write an equation for the line tangent to the solution curve that passes through the point $(1, 2)$.

(c) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(3) = -1$, and state its domain.

(a)



$2 : \begin{cases} 1 : \text{solution curve through } (3, -1) \\ 1 : \text{solution curve through } (1, 2) \end{cases}$

Curves must go through the indicated points, follow the given slope lines, and extend to the boundary of the slope field or the x-axis.

(b) $\left.\dfrac{dy}{dx}\right|_{(1,\,2)} = \dfrac{1}{2}$

An equation for the line tangent to the solution curve is $y - 2 = \dfrac{1}{2}(x - 1)$.

1 : equation of tangent line

(c) $y\,dy = x\,dx$

$\dfrac{1}{2}y^2 = \dfrac{1}{2}x^2 + A$

$y^2 = x^2 + C$

$C = -8$

Since the particular solution goes through $(3, -1)$, y must be negative.

$y = -\sqrt{x^2 - 8}$ for $x > \sqrt{8}$

$6 : \begin{cases} 5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases} \\ \text{Note: max } 2/5 \text{ [1-1-0-0-0] if no constant of integration} \\ 1 : \text{domain} \end{cases}$

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Question 6

Let $g(x) = xe^{-x} + be^{-x}$, where b is a positive constant.

(a) Find $\lim\limits_{x\to\infty} g(x)$.

(b) For what positive value of b does g have an absolute maximum at $x = \frac{2}{3}$? Justify your answer.

(c) Find all values of b, if any, for which the graph of g has a point of inflection on the interval $0 < x < \infty$. Justify your answer.

(a) $\lim\limits_{x\to\infty} g(x) = 0$	1 : answer
(b) $g'(x) = e^{-x} - xe^{-x} - be^{-x} = (1 - x - b)e^{-x}$ $g'\left(\frac{2}{3}\right) = \left(\frac{1}{3} - b\right)e^{-2/3} = 0 \Rightarrow b = \frac{1}{3}$ When $b = \frac{1}{3}$, $g'(x) = \left(\frac{2}{3} - x\right)e^{-x}$. For $x < \frac{2}{3}$, $g'(x) > 0$ and for $x > \frac{2}{3}$, $g'(x) < 0$. Therefore, when $b = \frac{1}{3}$, g has an absolute maximum at $x = \frac{2}{3}$.	$4 : \begin{cases} 2 : g'(x) \\ 1 : \text{solves } g'\left(\frac{2}{3}\right) = 0 \text{ for } b \\ 1 : \text{justification} \end{cases}$
(c) $g''(x) = -e^{-x} - (1 - x - b)e^{-x} = (x - 2 + b)e^{-x}$ If $0 < b < 2$, then $g''(x)$ will change sign at $x = 2 - b > 0$. Therefore, the graph of g will have a point of inflection on the interval $0 < x < \infty$ when $0 < b < 2$.	$4 : \begin{cases} 2 : g''(x) \\ 1 : \text{interval for } b \\ 1 : \text{justification} \end{cases}$